Logic: As Simple as A Implies B?

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What is Logic?

Wikipedia says:

"Logic is the formal systematic study of the principles of valid inference and correct reasoning. Logic is used in most intellectual activities, but is studied primarily in the disciplines of philosophy, mathematics, semantics, and computer science. Logic examines general forms which arguments may take, which forms are valid, and which are fallacies. In philosophy, the study of logic figures in most major areas: epistemology, ethics, metaphysics. In mathematics, it is the study of valid inferences within some formal language."

1. In daily life: Most intellectual activities
   - Logic puzzles - SUDOKU or THE ONE-QUESTION HERO
   - Games - MASTERMIND or SET
2. In mathematics: Valid inferences within some formal language.

Logic Puzzles

THE ONE-QUESTION HERO OF LOGIC

Logan the Logician is in a dark forest on a quest to save the kingdom from fanatical fallacy when he comes to a fork in the road. One path leads to the Abyss of Logical Contradiction and Headache, while one path leads to the Palace of Sensible Thinking. Two men stand before him, one of which always lies and one of which always tells the truth. However, Logan does not know which path is which, or which man is which.

What single question can he ask one of the men to deduce the way to the palace?

Mathematical Logic: Truth Tables

Math Logic is valid inferences within some formal system

Table: Common Truth Table Symbols

Symbol	Meaning
$\neg$	not (negation)
$\wedge$	and (conjunction)
$\vee$	or (disjunction)

Logic Games

Logic examines general forms which arguments may take, which forms are valid, and which are fallacies.

I Challenge You to

MASTERMIND !

Mathematical Logic: Deeper Results

Math Logic is valid inferences within some formal system

1. Is the following statement true?

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In fact, Kurt Gödel proved in 1931 that within formal systems that contain arithmetic, there are true statements that cannot be proved!

Kurt Gödel, Logician

Gödel's Proof Overview

1. Construct G, an arithmetical formula that represents the meta-mathematical statement, "The formula G is not demonstrable"
2. Prove G is demonstrable if and only if $\sim G$ is demonstrable. Inconsistent!
3. Show G is true.
4. Thus arithmetic is incomplete (there are true statements that cannot be proved).

Gödel's Formal System Details

1. Gödel numbering
   - Symbols of formal system given a unique number
   - **Example:** # of 0 is 6, # of $=$ is 5, # of y is 19

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2. $Dem(x, z)$
   "Formula with number x is proof of formula with number z"
3. $sub(y, 19, y)$
   "Gödel number of the formula resulting from: Replace all y with the number y in equation with number y"

Gödel's Formula G

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$$(x) \sim Dem(x, sub(y, 19, y))$$

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- Meta-mathematical meaning of G?

"The formula G,
$(x) \sim Dem(x, sub(n, 19, n))$,
is not demonstrable"

Gödel's Conclusion

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Gödel concluded:

- If G is demonstrable then $\sim G$ is demonstrable
- But then arithmetic is inconsistent
  (cannot have a statement and its negation simultaneously true)
- Thus, assuming consistency, G is not demonstrable, and thereby a true statement

In other words, there exist true statement(s) within all consistent formal systems that cannot be proved! Arithmetic is incomplete!

Our time is up!

Enjoy the rest of Math Circles 2011!